INVESTIGATION OF FREQUENCY RATIOS

- Vertical oscillation frequency, ν
- Poisson Eq.

$$4\pi G\rho = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2}$$
$$\simeq \frac{1}{R}\frac{\mathrm{d}v_c^2}{\mathrm{d}R} + \nu^2$$

- Disk with flat $v_c = \text{constant} \to \boxed{\nu^2 = 4\pi G\rho}$
- Flat $v_c \Rightarrow \kappa^2 = 2\Omega^2$, also $\Omega^2 \approx \frac{GM}{R^3} = \frac{4}{3}\pi G\bar{\rho}$. Roughly true even for a disk $\Rightarrow \frac{\nu^2}{\kappa^2} = \frac{\nu^2}{2\Omega^2} = \frac{4\pi G\rho}{\frac{8}{3}\pi G\bar{\rho}} = \frac{3}{2}\frac{\rho}{\bar{\rho}}$ measures the degree to which mass is concentrated towards the plane.
- Table 1.1 $\quad\rho \approx 0.1\mathrm{M_\odot pc^{-3}}$ near the Sun
- Vertical oscillation period $\frac{2\pi}{\nu} = \frac{2\pi}{\sqrt{4\pi G\rho}} = \boxed{87\ \mathrm{Myr}}$
- $T = \frac{2\pi}{\Omega} = \frac{2\pi R_0}{v_0} = \frac{8.2\ \mathrm{kpc}}{240\ \mathrm{km/s}}\cdot 2\pi = \frac{8200\ \mathrm{pc}}{240\ \mathrm{pc/Myr}}\cdot 2\pi = \boxed{210\ \mathrm{Myr}}$
  $1\mathrm{km} = 1.023\ \mathrm{pc/Myr}$
- $\frac{2\pi}{\kappa} \approx \boxed{155\ \mathrm{Myr}}$ if $\frac{\kappa_0}{\Omega_0} = 2\sqrt{\frac{-13}{A-B}} = 1.35$
- $\frac{\nu}{\kappa} \approx 1.8$ for the Sun $\Rightarrow \bar{\rho} = \frac{3}{2}\rho\cdot\left(\frac{\kappa}{\nu}\right)^2 = 0.046\ \mathrm{M_\odot pc^{-3}}$
- Harvard professor Lisa Randall wrote a book "How dark matter killed dinosaurs". Massive extinction period $\sim 30\mathrm{Myr}$? Dinosaurs, $\sim 66\ \mathrm{Myr}$ ago. Dissipating DM particle $\to$ DM disk, $\Sigma = 10\ \mathrm{M_\odot/pc^2}$, $z_d = 10\ \mathrm{pc}$
  $\nu = \sqrt{4\pi G\rho}\quad \rho_{\mathrm{DM}} \sim 1\mathrm{M_\odot}/pc^3$
  $\to \nu$ is increased by a factor 3 ?
  $\to \frac{2\pi}{\nu} \to 30\mathrm{Myr}?$

THE THIRD INTEGRAL OF MOTION

- General orbits in an axisymmetric potential E and L_z are integrals of motion, but are there more? The two orbits with the same E and L_z, but they look very different! Difference does not diminish, no matter how long they are integrated.

  $\rightarrow$ third integral?

- Eq. (3.70) $\Phi_{\rm eff} = \frac{1}{2} v_0^2 \ln \left(R^2 + \frac{Z^2}{q^2} \right) + \frac{L_z^2}{2R^2}$

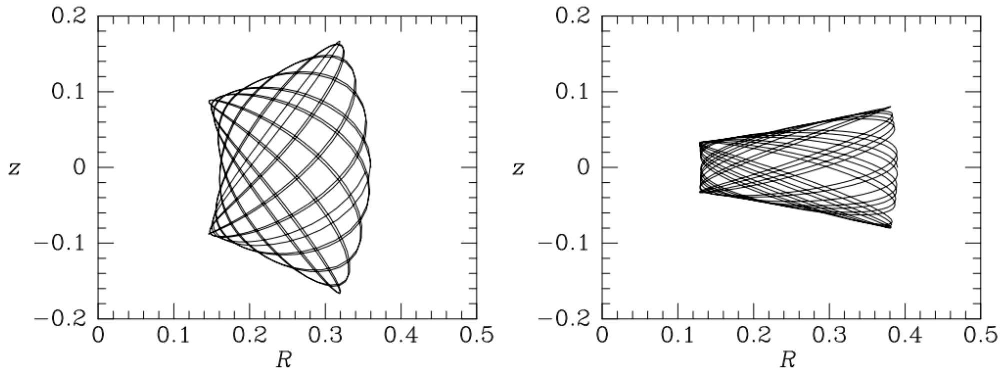

Figure 3.4 Two orbits in the potential of equation (3.70) with $q = 0.9$. Both orbits are at energy $E = -0.8$ and angular momentum $L_z = 0.2$, and we assume $v_0 = 1$.

- $\ddot{R} = -\frac{\partial \Phi_{\rm eff}}{\partial R}$ $\quad \ddot{z} = -\frac{\partial \Phi_{\rm eff}}{\partial z}$ used $L_z = \text{constant}$ to reduce motion in meridional plane, (R, z)

- How to visualise. 4-D phase space $(R, \dot{R}, z, z)$?

  $H_{\rm eff}(R, z, \dot{R}, \dot{z}) = \text{constant} = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{\rm eff} \rightarrow 3D$

- Poincoré surface of section (SOS): cut 3D ellipsoidal volume in $(R, z, \dot{R})$, construct a SOS diagram to show the phase space in 2D subspace $(R, \dot{R})$

- $z = 0$, and $\dot{z} > 0$ (moving upwards), record $(R, \dot{R})$ consequences to remove sign ambiguity. $\rightarrow$ no distinct orbits at the same E can occupy the same point.

- Zero velocity curve (zvc) in sos. ($\dot{z} = 0$)

$$H_{\rm eff} \geqslant \frac{1}{2}\dot{R}^2 + \Phi_{\rm eff}(R, z = 0)$$

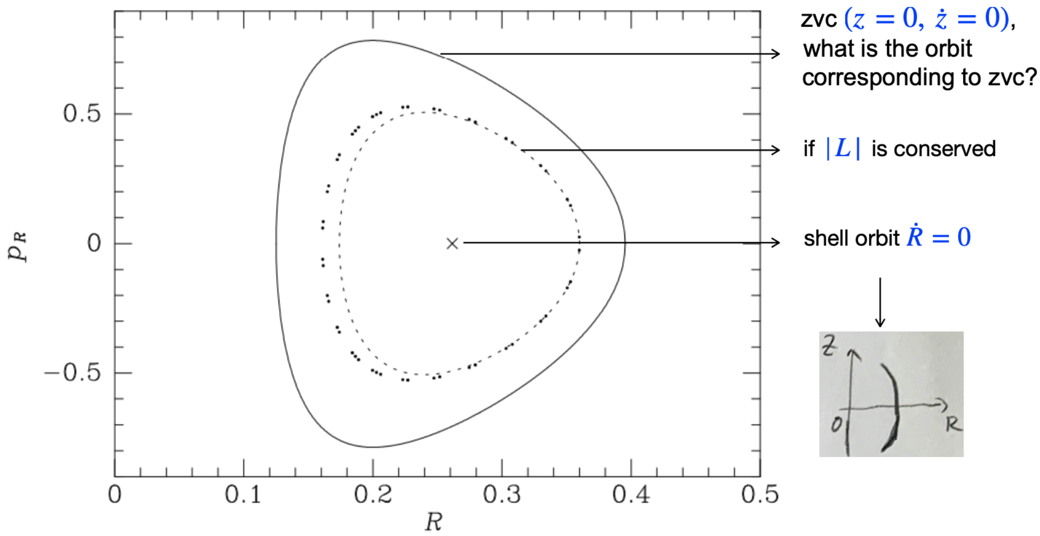

Figure 3.5 Points generated by the orbit of the left panel of Figure 3.4 in the (R, p_R) surface of section. If the total angular momentum L of the orbit were conserved, the points would fall on the dashed curve. The full curve is the zero-velocity curve at the energy of this orbit. The $\times$ marks the consequent of the shell orbit.

- If I_3 exists, orbits lie on a smooth curve: Invariant curve (1-D curve in 2-D space), otherwise can fill up the area inside zvc.
- $I_3 =$ non-classical integral, because I_3 has no analytical expression in $(\vec{x}, \vec{v})$
- We may get an intuitive picture of the nature of $I_3 =$ by considering two special cases:

  ① since $|L| =$ integral for a spherical potential so for a nearly spherical potential. $I_3 \approx |L|$ see the dashed line in Fig. 3.5.
  $|L(t)|$ oscillates rapidly, but its mean value does not change. So $|L|$ is an approximately conserved quantity, even in a flattened potential. Orbits are approximately planar with r_{peri} and r_{apo}. The approximate orbital plane has a fixed inclination to the z-axis but precesses about z. Precession rate $\to$ 0, for a increasingly spherical.

  ② Potential separable in R and z

$$\Phi(R, z) = \Phi_R(R) + \Phi_z(z)$$

  Then I_3 can be $H_Z = \frac{1}{2}p_z^2 + \Phi_z(z)$ In the case of epicycle approximation, what is the shape of the invariant curves?

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THE SCHWARZSCHILD DISTRIBUTION OF STARS IN THE MILKY WAY DISK

- Gas : Every component of the velocity distribution v_i follows a **Gaussian** probability distribution.

$$f_D\left(v_i\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{v_i^2}{2\sigma z}\right]$$

where $v_i = v_x, v_y, v_z$, $\sigma_x = \sigma_y = \sigma_z = \sigma$ **isotropic**

$$f(\vec{v})\,\mathrm{d}^3v = f_D\left(v_x\right) f_D\left(v_y\right) f_D\left(v_z\right) \mathrm{d}^3v = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 \exp\left[-\frac{|v^2|}{2\sigma^2}\right] \mathrm{d}^3v$$

where $|v|^2 = v_x^2 + v_y^2 + y_z^2$

  - $v \to v + \mathrm{d}v$

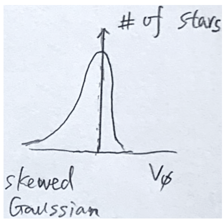
$$f(\vec{v})\,\mathrm{d}^3v = f_M(v)\,\mathrm{d}^3v = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 4\pi v^2 \exp\left(-\frac{v^2}{2\sigma^2}\right) \mathrm{d}v \leftarrow \text{Maxwellian DF}$$

- Stars: velocity dispersion is different for every direction **anisotropic**

- Schwarzschild first postulated that the probability of (v_R, v_ϕ, v_z) in d^3v is

$$P(\vec{v})\,\mathrm{d}^3v = \frac{\mathrm{d}^3v}{(2\pi)^{3/2}\sigma_R\sigma_\phi\sigma_z} \exp\left[-\left(\frac{v_R^2}{2\sigma_R^2} + \frac{v_\phi^2}{2\sigma_\phi^2} + \frac{v_z^2}{2\sigma_z^2}\right)\right]$$

- But it fails to reproduce the asymmetrical distribution of v_ϕ for stars with a higher random velocity. How do we explain it?



- Stars in a galactic disk travel on nearly circular and coplanar orbits. Goal: find a DF that generates cool disks in which random velocities are much smaller than v_c

- The mean radius (or guiding radius) of nearly circular orbits:

$$F_R = \left(\frac{\partial\Phi}{\partial R}\right)_{R=R_g} = \frac{v_c^2}{R_g} = \frac{L_z^2}{R_g^3} \qquad \text{(Epicycle approximation)}$$

- $L_z = R_g^{\frac{3}{2}} \cdot \left(\frac{\partial \Phi}{\partial R}\right)_{R_g}^{1/2} = L_z\left(R_g\right) \quad \leftrightarrow \quad R_g = R_g\left(L_z\right)$, one to -one relation.
  - Note $\frac{\partial L_z}{\partial R} \geqslant 0$ for stable circular orbits.
- Thus the radial density profile $\Sigma(R)$ of a cool disk (low velocity dispersion) is largely determined by the dependence of the DF upon L_z.

$$\Delta \equiv H - E_c(L_z)$$

where $E_c\left(L_z\right)$: energy of a circular orbit with L_z,

$$E_c(R) = \frac{1}{2}v_\phi^2 + \Phi(R) = \frac{1}{2}\frac{L_z^2}{R^2} + \Phi(R) \xlongequal{R=R(L_z)} E_c\left(L_z\right)$$

Δ is the energy associated with the random motion around the guiding center.

- Many stars with epicyclic oscillations in random phases lead to a velocity